BAYESIAN OPTIMIZATION FOR RAPID PROBABILISTIC ESTIMATIONS OF OVERALL LEVEL ON FREQUENCY RESPONSE MODELS

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ABSTRACT

Traditionally, frequency response models are solved using either a constant bandwidth or an n-th octave band. For both vibration and acoustic analyses, it is commonly known that one is expected to solve a model with a sufficiently narrow band to ensure the overall response of the mechanical system is accurately captured. However, when specifically capturing the overall level, one needs to ensure the highest values of the response curve are correctly captured while a controlled amount of uncertainty can be accepted when estimating the lowest response point of the curve.

We propose a Bayesian optimization technique designed to capture the overall level using a Matern kernel. The proposed method enables estimations of the overall levels of a response curve using approximately five times fewer frequency points while providing a given uncertainty on the overall level.

This paper describes the architecture of the proposed process, and demonstrates validation using individual response curves followed by an integrated process with simulation software.

INTRODUCTION

Accurate frequency response models are crucial in vibration and acoustic analyses. These models require the precise capture of system responses over specific frequency bands to ensure reliable predictions. Traditional methods often involve extensive computations, which can be resource-intensive and time-consuming. Such approaches typically involve solving models using either constant bandwidths or n-th octave bands with frequency steps that are fine enough to ensure an accurate representation of the overall response of mechanical systems.

However, when focusing only on the overall level, the basic frequency domain definitions can be revised. One can choose to accurately capture the highest response values while tolerating some uncertainty in estimating the lowest response points. This generally leads to a reasonably accurate capture of the overall level.

Bayesian optimization presents a promising technique to address these challenges. Specifically, by utilizing an appropriate kernel and tuned decision algorithm within a Bayesian framework, it is feasible to estimate overall response levels effectively with significantly fewer frequency points, balancing computational efficiency with the accuracy provided by probabilistic estimations. Bayesian methods, known for their robust statistical inference and parameter estimation capabilities, have been extensively applied in various engineering problems, including model updating and uncertainty quantification [1]. The utilization of Bayesian optimization, particularly with Gaussian process (GP) surrogate models, has shown efficacy in achieving rapid and precise optimization in high-dimensional spaces [2].

In the context of frequency response models, Bayesian optimization facilitates the construction of a probabilistic model of the objective function, enabling efficient exploration and exploitation of the search space to identify optimal frequency. This approach not only reduces the computational burden but also enhances the robustness of the estimations by integrating prior knowledge and observed data into the model [3].

By focusing on capturing the highest response values accurately while accepting controlled uncertainty in the lower response points, the method proposed in this paper achieves a balance between accuracy and computational efficiency. This is particularly advantageous in practical applications where computational resources and time are often constrained.

The proposed method involves constructing a surrogate model using Gaussian processes, which is then optimized using Bayesian techniques to identify the most informative frequency points. This approach reduces the required frequency points by approximately five times compared to traditional methods, thereby significantly enhancing the efficiency of the modelling process [4].

To validate the proposed method, individual response curves are analyzed, followed by integration with simulation software and a coupled “finite element, boundary element method (FE-BEM)” to assess its performance in practical scenarios. The results indicate that the proposed Bayesian optimization approach not only maintains high accuracy in capturing overall response levels but also offers substantial improvements in computational efficiency.

METHODOLOGY

Bayesian optimization is selected due to its robustness in handling complex, noisy functions with limited data points. Initially, a standard Radial Basis Function (RBF) kernel was considered for the Gaussian process model. The RBF kernel, also known as the squared exponential kernel, is defined by:

$$k_{RBF}(d) = -\exp\left(-\frac{d^2}{2}\right)^{\nu}$$

where:

- $d = \frac{|f - f'|}{l}$ is the distance between two frequency points f and f′,
- l is the length scale parameter. [5]

While the RBF kernel is effective for very smooth functions, it was found to be unsuitable for this application due to its tendency to overly smooth the response curves, which do not reflect the true nature of vibration and acoustic signals characterized by abrupt changes and peaks.

To address this, the Matern kernel was chosen [5]. The Matern kernel, a popular choice in Gaussian process modeling, is employed for its flexibility and ability to model various degrees of smoothness in the response curves. This kernel is particularly effective in scenarios where the response function is not smooth but rather characterized by abrupt changes or discontinuities. Additionally, the Matern kernel allows for peaks in the curve, which works well for capturing the sharp peaks often observed in frequency response functions.

The Matern kernel covariance function is defined as:

$$k_{\nu}(d) = \frac{2^{1-\nu}}{\Gamma(\nu)}\left(\sqrt{\{2\nu\}}d\right)^{\nu} K_{\nu}\left(\sqrt{\{2\nu\}}d\right)$$

- ν controls the smoothness of the function,
- $\Gamma(\nu)$ is the Gamma function,
- K_{ν} is the modified Bessel function of the second kind.

Constructing the Surrogate Model

The first step in the proposed methodology involves constructing a surrogate model using Gaussian processes (GPs). Gaussian processes are non-parametric models that provide a probabilistic approach to modeling complex functions [6]. The surrogate model approximates the true response function, capturing its key characteristics while being computationally less demanding.

The process begins by selecting a set of initial frequency points, which are used to evaluate the response function. These points are chosen based on prior knowledge or through a preliminary experimental design. In the examples described in this paper, we found that the one third octave band provides that satisfactory initial data set; however, this may need to be adjusted for different models. The response values at these points serve as the training data for the GP model. The Matern kernel is then used to define the covariance structure of the GP, enabling the model to capture the dependencies between different frequency points.

Optimization Using Bayesian Techniques

Once the surrogate model is established, Bayesian optimization is employed to identify the most informative frequency points that need to be evaluated to estimate the overall level accurately.

As a reminder, the overall level is defined as the integral of the PSD vibration response of the structure excited by a diffuse acoustic field [7] . For a given model it is estimated by:

$$O.L. = \sqrt{\sum_{n} S_n^{(psd)} \Delta f}$$

This can be rewritten as:

$$O.L. = \int_{\text{start band}}^{\text{end band}} S_n^{(psd)}(f)\mathrm{d}f$$

The main idea behind this optimization process is to estimate this overall level by minimizing the number of frequency points.

The Bayesian optimization process involves the following steps:

1. **Acquisition Function Selection**: An acquisition function is chosen to guide the search for the next frequency point to evaluate. The acquisition function balances exploration (sampling points with high uncertainty) and exploitation (sampling points with high expected response). This part of the algorithm requires the most adaptation to estimate the overall level of the response while reducing the uncertainty of the overall response in the

model. A combination of exploitation and exploration is used, formulated as follows:

$$\alpha(f) = \sigma(f) \cdot (\mu(f) + \sigma(f))$$

where:

- $\sigma(f)$is the uncertainty (standard deviation) at frequency f,
- $\mu(f)$ is the predicted mean level at frequency f.

The next frequency point to be solved is the frequency f where α is maximum. As the frequency point is solved, $\sigma(f)$ tends to zero, this $\alpha(f)$ tends to zero and the next frequency solved is a combination of levels and uncertainty. This was found to best identify the overall level of the function while minimizing the number of frequency points.

2. **Iterative Updating**: The surrogate model is iteratively updated with new frequency points. At each iteration, the acquisition function is maximized to select the next frequency point, which is then evaluated using the true response function. This new data point is then added to the training set, and the GP model is updated accordingly.

3. **Convergence Criteria**: The optimization process continues until a convergence criterion is met. This criterion can be based on a predefined number of iterations, a threshold on the acquisition function value, or a desired level of uncertainty in the overall level estimation. In our model, we calculate the overall level for the vibration level at a given target location and continue to convergence. In the experiments presented below, we defined convergence on a stable solution as an overall level change by less than 1% when adding 10 new frequency points.

Reducing the Number of Frequency Points

A key advantage of the proposed methodology is its ability to reduce the required number of frequency points by approximately five times compared to traditional methods. This is illustrated by Figure 3 where both the original function (orange) and the approximated function (blue) are shown. We see here that the function can easily be approximated with a reduced number of points. This reduction is achieved through the efficient sampling strategy of Bayesian optimization, which focuses on the most informative points rather than uniformly sampling the entire frequency range. This approach not only decreases the computational burden but also maintains the accuracy of the overall level estimation. Given a reasonable number of frequency points on the initial surrogate model, the estimation of the overall level can be obtained through an iterative gaussian optimization process.

Practical Applications and Advantages

The proposed methodology is particularly advantageous for frequency response models where computational resources and time are constrained. For example, in vibration response predictions to structural or acoustic loads, where frequent updates to the frequency response are needed for the different design iterations, reducing the number of frequency points can significantly speed up the analysis without compromising accuracy. Additionally, this method can be applied to acoustic analyses of large mechanical systems such as launch vehicles, allowing for quick and reliable estimations of overall sound pressure levels while controlling uncertainties, ultimately facilitating timely decision-making.

The proposed method can be applied to coupled models, include those that are strictly finite element based or those that combine the finite element method with others including boundary element methods and statistical analysis.

By integrating Bayesian optimization with a Matern kernel to create a smart acquisition function, this methodology offers a robust and efficient solution for estimating the overall levels of frequency response models, balancing the need for accuracy with computational efficiency.

VALIDATION AND APPLICATIONS

Individual frequency response curve

To validate the proposed method we utilize a known frequency response curve with a defined overall level to create a surrogate function. The optimization algorithm is then run until the known overall level is reached.

The initial function is a vibration response is shown in Figure 1 with an overall level of 61.5 g rms that uses 253 frequency points from 20 to 2000 Hz in the 1/36th Octave Band. Given the response curve and the level of damping on the structure, sampling the curve with narrower frequency bands should not affect the overall level. For convenience, all the results presented below use a Power Spectral Density (PSD) in g^2/Hz.

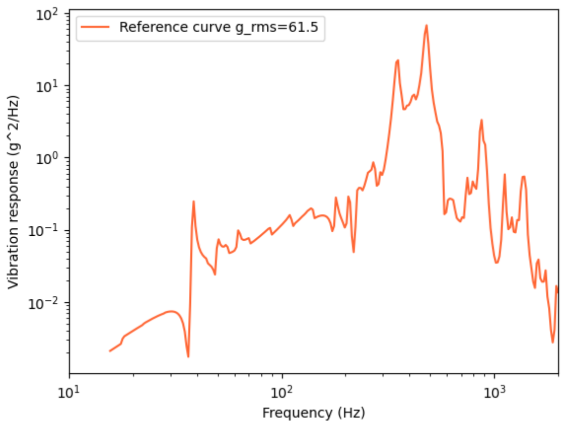

Figure 1: PSD vibration response curve

As presented in the previous section an initial surrogate function is initially approximated using 21 frequency points in the one third octave band. The above frequency response is first approximated with the following curve:

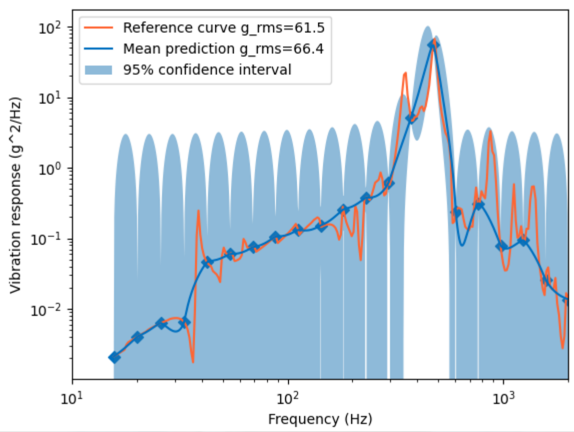

Figure 2: PSD Vibration curve and initial surrogate function in the one third octave band with 95% confidence interval

The confidence interval looks unusual due to the logarithmic scale of the graph. At sample points, uncertainty goes to zero. Model parameters appear to be well tuned, as the reference curve is contained within a 95% confidence interval.

Note that this paper does not discuss the determination of the length scale and ν parameters. These were determined using multiple vibration response curves and are set to $l = 0.671$ and $\nu = 0.75$. Values were determined using a publicly available optimization algorithm in the Python library *sklearn* [8].

After parameter selection using the described optimization process, the proposed acquisition function is then used to determine additional frequencies at which the response function should be sampled. Figure 3 shows the initial curve (orange), augmented by 31 additional frequency points where the convergence criteria is met.

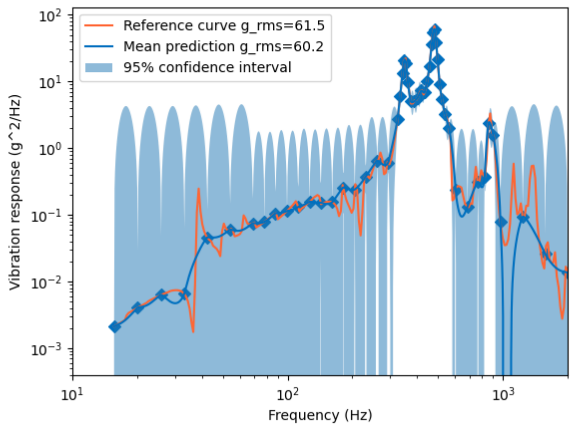

Figure 3: PSD Vibration curve and optimized surrogate function with 95% confidence interval

Figure 4 shows that the estimation error decreases as additional frequency points are added to the surrogate function.

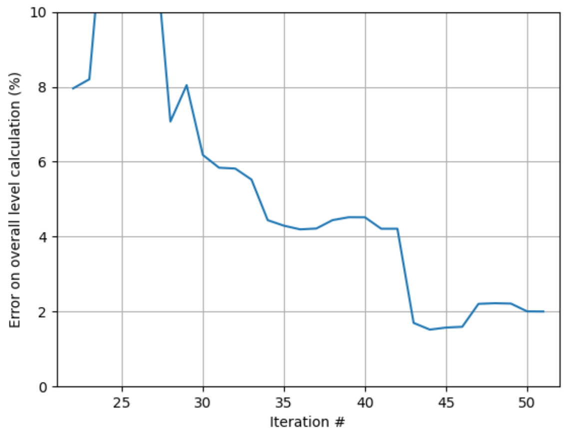

Figure 4: Error on overall level as frequency points are added to the surrogate function

To test the algorithm stability, the Bayesian optimization algorithm is run to include up to 100 frequency points, resulting in the response curve shown in Figure 5 and corresponding error plot shown in Figure 6.

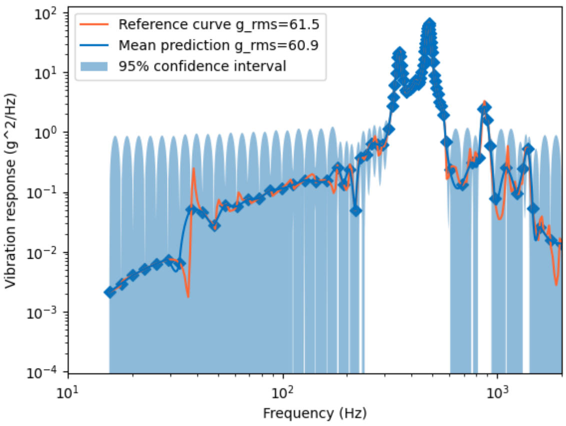

Figure 5: Second PSD Vibration curve and optimized surrogate function with 95% confidence interval

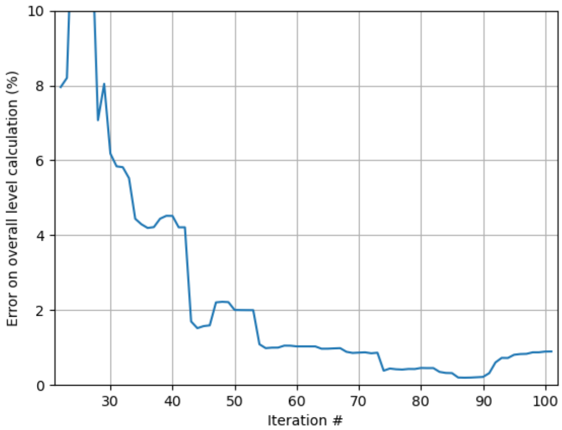

Figure 6: second Error on overall level as frequency points are added to the surrogate function

Although the error on the overall level oscillates between 0 and 1% over a number of iterations, the algorithm is generally stable as all of the data recovery locations we selected led to a converged value. However, before extensive use in an industrial simulation, additional robustness validation should be performed. Nevertheless, for the purpose of this paper, this criterion can be used to estimate overall levels. This stability confirms the proposed convergence criteria described previously.

We next applied the model to additional response curves (Fig 7, Fig. 9).

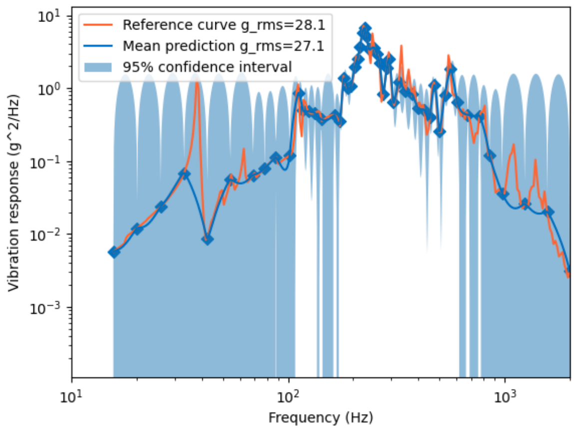

Figure 7: Third PSD Vibration curve and optimized surrogate function with 95% confidence interval

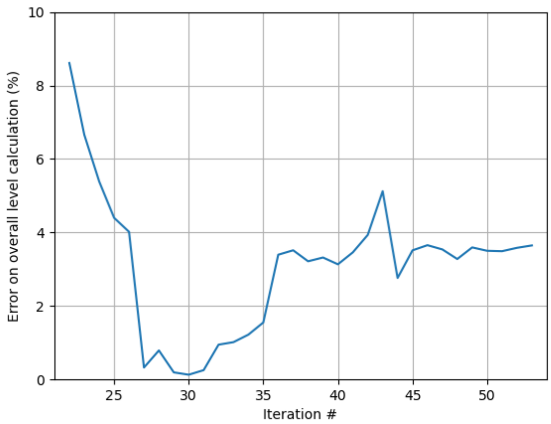

Figure 8: Third error on overall level as frequency points are added to the surrogate function

For the experiment shown in Fig. 7, It is worth noting that the error graph is the lowest with the 30th frequency point, despite the fact that the optimization has not converged, which occurs after approximately 50 frequency points and the error on the overall level is below 4%; this could be explained by a large peak at 50 Hz is not accounted for. However, the error can still be considered minimal. Additional tests with stricter convergence criteria showed that all peaks can be captured by the proposed algorithm (Fig. 9). However, a 4% error on the estimation of the overall level can generally be deemed acceptable on simulation models, as other sources of error exist between simulation and test.

Another standard response curve showing the satisfactory performance of the proposed algorithm is shown below.

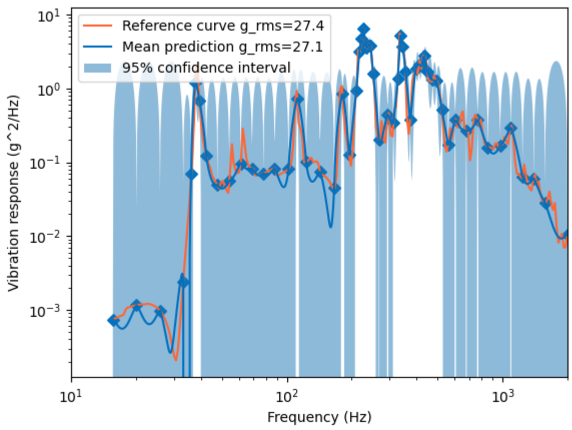

Figure 9: Fourth PSD Vibration curve and optimized surrogate function with 95% confidence interval

The error graph is:

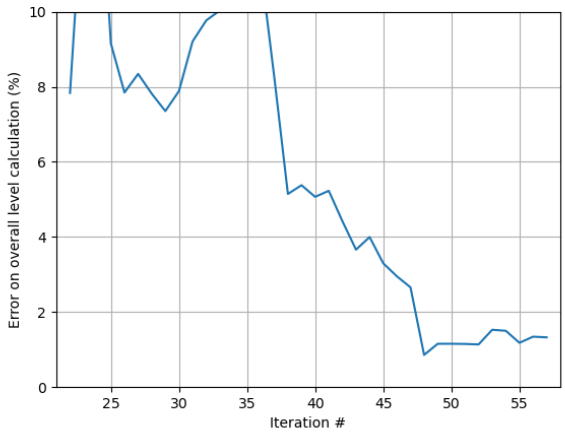

Figure 10: Fourth error on overall level as frequency points are added to the surrogate function

When considering the 95% confidence interval, assuming that the error at a given frequency is uncorrelated to the error at another frequency, we can calculate a standard deviation for the overall level using the following formula:

$$\sigma_{overall} = \sqrt{\sum \sigma(f_i)^2}$$

For five consecutive curves, we are the comparing the overall level, of the overall standard deviation and the residual error:

Curve #	Overall level [g rms]	$\sigma_{overall}$ [g rms]	Residual error %
1	60.2	1.0	2.0
2	55.4	1.1	1.2
3	27.1	0.8	3.6
4	27.1	0.8	1.3
5	8.5	0.3	0.4

The values above show the consistency of the algorithm across several curves. The maximum overall standard deviation remains low and therefore the calculated overall level is reliable, suggesting that this method can be used in an engineering study.

For the 5 curves studied in this section, the maximum number of frequency points to reach convergence is 56. Considering the data source is comprised of 253 frequency points, this means that we can correctly approximate the response curve with almost 1/5th of the frequency points. On a frequency response calculation, this should theoretically translate into a nearly 5x speedup on the computation time.

Integration with a coupled FE-BEM model

A coupled Finite Element-Boundary Element Model (FE-BEM) in VA One. The model applies a Diffuse Acoustic Field modeled as a sum of individual plane waves. This type of model is standard for most payload analysis to evaluate the vibration and stresses.

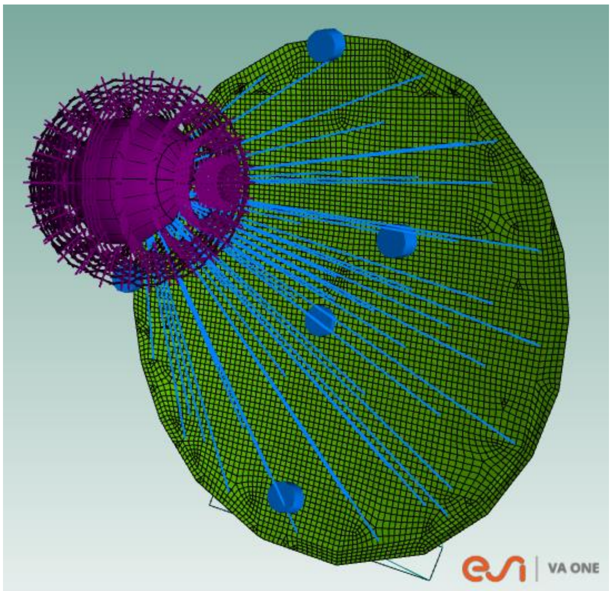

Figure 11: VA One FE-BEM Model render

For this model, we aim to predict the overall vibration response at 6 locations from 20 to 500 Hz. The structural part of the model functions as a modal frequency response. Structural modes are calculated up to 720 Hz. Both the BEM and the coupled model are solved for each individual frequency requested by the algorithm.

The algorithm was modified so that a single simulation model is used to predict the responses at 6 individual locations. For this, an individual surrogate function is generated for each of the 6 recovery locations. Then, for the optimization process, the highest value of the acquisition function determines the next frequency to be solved, though each new frequency point feeds all 6 surrogate functions. The model is considered to be solved once all 6 surrogate functions meet the convergence criterium previously defined.

After running the algorithm, convergence was reached on all 6 sensors after 68 iterations.

Structural response results are presented in the figure below:

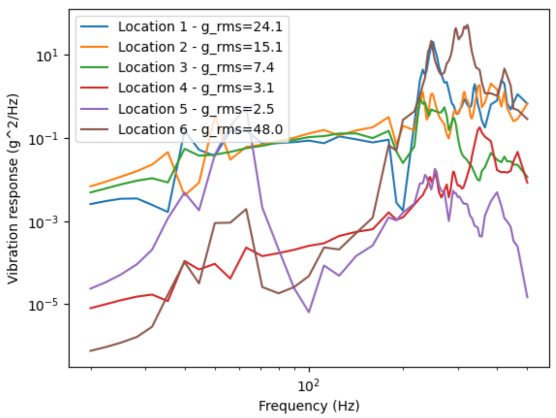

Figure 12: Vibration response at 6 locations on the VA One FE-BEM Model

When compared to a traditional model solved in the 1/36th Octave band the results compare as follows:

Curve	Bayesian optimization overall level	1/36th Octave band solve overall level	Difference (%)
1	24.1	24.3	0.8%
2	15.1	15.8	4.4%
3	7.4	7.4	0.0%
4	3.1	3.1	0.0%
5	2.5	1.8	-38.9%
6	48.0	48.2	0.4%

Convergence curves are presented in Figure 13:

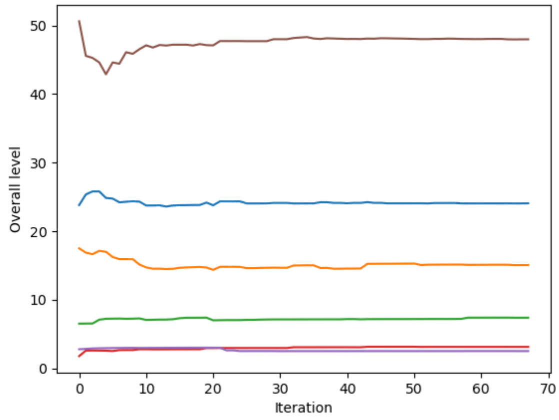

Figure 13: Overall level for the 6 sensor locations as additional frequency points are added to the data set.

Note that the model is considered to be solved when the convergence criteria is met on all 6 curves. When looking at the difference of overall level between the Bayesian optimization and the original 1/36th Octave band function, we see that the error is minimal on function where the overall level is relatively high. For functions where the response is low, as seen in curve 5, we observe the large difference as the acquisition function privileges higher responses (38.9%). Should the uncertainty on those lower responses be of importance to the studied structure, one may choose to modify the acquisition function or consider this response location individually.

Here, we see that the algorithm converges quickly as all 6 surrogate functions get populated when a new frequency point is solved. This results in a significant reduction in the required number of frequency points, thus solve time.

CONCLUSIONS

This research introduces a novel Bayesian optimization method using a Matern kernel, significantly enhancing computational efficiency while maintaining high accuracy in frequency response analysis. This approach is particularly beneficial in scenarios where computational resources are limited. By addressing the inefficiencies of traditional methods, the proposed technique significantly enhances computational efficiency while maintaining high accuracy in estimating overall response levels. This approach is particularly beneficial in scenarios where computational resources and time are limited, such as in structural vibration predictions and acoustic analysis.

Key Contributions

1. **Efficiency and Accuracy**: The proposed Bayesian optimization method reduces the number of frequency points required by approximately five times compared to traditional methods. This reduction is achieved through the efficient sampling strategy of Bayesian optimization, focusing on the most informative points. The method balances the need for computational efficiency with the accuracy of the probabilistic estimations, which is critical for practical engineering applications.
2. **Relevance of the Matern Kernel**: The selection of the Matern kernel over the RBF kernel is validated through its ability to handle abrupt changes and discontinuities in the response curves. This makes the methodology well-suited for modeling real-world vibration and acoustic signals, which are often characterized by sharp peaks and non-smooth behavior.
3. **Scalability and Adaptability**: The Bayesian optimization approach is scalable and adaptable to various types of response curves and models. This is demonstrated through the integration of the methodology with simulation software and coupled FE-BEM. The ability to apply the method across different models and scenarios showcases its versatility and potential for broader application in engineering analyses.
4. **Validation and Practical Application**: The validation of the methodology through known frequency response curves and its application to a coupled FE-BEM model highlights its practical utility. The results show that the Bayesian optimization approach maintains high accuracy in capturing overall response levels while offering substantial improvements in computational efficiency. This practical validation confirms the method's reliability and effectiveness in real-world applications.

Future Work

The promising results of this research open several avenues for future work:

1. **Enhanced Kernel Selection**: Future research could explore the potential of other kernels or hybrid kernel approaches to further enhance the accuracy and efficiency of the Bayesian optimization method. Tailoring the kernel to specific types of response curves could provide even better performance.
2. **Automated Parameter Tuning**: Developing automated techniques for tuning the hyperparameters of the Gaussian process model, such as the length scale and ν parameters, could further improve the robustness and adaptability of the methodology. This would make the approach more user-friendly and applicable to a wider range of problems.
3. **Integration with Commercial Software Packages**: The last example presented in this paper demonstrates a potential integration with a software package. Integrating the Bayesian optimization approach with convergence monitoring could provide significant benefits in applications where approximate level can first be determined and accuracy is increasing as additional frequency points are added. This not only would lower the overall solve time, but would also allow the user to get an initial approximate answer while the calculation is still converging.
4. **Extended Applications**: Exploring the application of the methodology to other domains, such as electromagnetic analysis or thermal response modeling, could demonstrate its versatility and effectiveness beyond vibration and acoustic analyses. Extending the approach to these areas could provide new insights and advancements in engineering simulations.

In conclusion, this research advances the state-of-the-art in Bayesian optimization for frequency response models, offering a robust and efficient solution for probabilistic estimations. The methodology's ability to balance accuracy with computational efficiency makes it a valuable tool for engineering analyses, paving the way for more reliable and high-fidelity models that can better predict and respond to real-world conditions.

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